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An Introduction to the Fractional Calculus: Theories and Applications
Continued…

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Geometrical Interpretation of fractional integrals

- The non-existence of a geometrical or physical interpretation of the fractional derivatives or integrals was acknowledged in the first world conference on Fractional Calculus and Applications held in 1974.
- F Ben Adda suggested in 1997 a geometrical interpretation using the idea of a contact of the α th order. But his interpretation did not contain any "pictures".
- Igor Podlubny in 2001 discovered an interesting geometric interpretation of fractional integrals based on the geometrical interpretation of the Stieltjes integral discoverd by G L bullock in 1988. In this talk we present Podlubny's geometric interpretation of the fractional integral.

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Geometrical interpretation of Stieltjes integral

Let $g(x)$ be a monotonically increasing function and let $f(x)$ be an arbitrary function. We consider the geometrical interpretation of the Stieltjes integral

$$\int_a^b f(x)\, dg(x).$$

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- Choose three mutually perpendicular axes : g-axis, x-axis, f-axis.
- Consider the graph of $g(x)$, for $x \in [a, b]$, in the (g, x)-plane. Call it the $g(x)$-curve.
- Form a *fence* along the $g(x)$-curve by erecting a line segment of height $f(x)$ at the point $(x, g(x))$ for every $x \in [a, b]$.
- Find the *shadow* of this fence in the (g, f)-plane.
- Area of the shadow is the value of the Stieltjes integral $\int_a^b f(x)\, dg(x)$.

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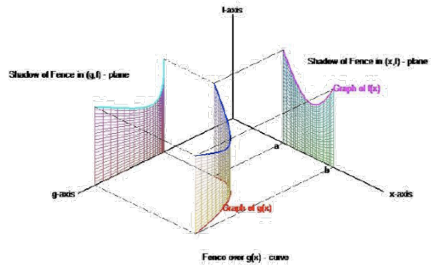

Area of shadow of fence in (g, f)-plane$=\int_a^b f(x)\,dg(x)$.

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Geometrical interpretation of fractional integral

- For $q < 0$ we have

$$_aD_x^q f(t) = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(t)}{(x-t)^{q+1}} dt.$$

- We write

$$g(t) = \frac{1}{\Gamma(-q+1)} \left[\frac{1}{x^q} - \frac{1}{(x-t)^q} \right]$$

- We have the Stieltjes integral

$$_aD_x^q f(t) = \int_a^x f(t)\, d\, g(t).$$

- This can be interpreted as the area of the shadow of a fence.

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In he next few frames we present the visualizations of the fractional integral

$$_0D_x^q(f(t))$$

when

$$f(t) = t + 0.5\sin(t)$$

for the following values of q :

$$q = -0.25, \quad -0.5, \quad -1, \quad -2.5$$

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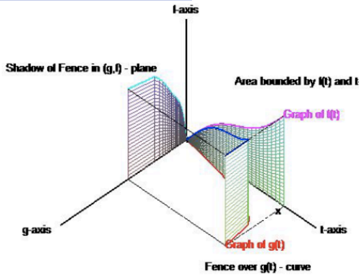
$_0D_x^{-0.25}(t + 0.5\sin(t))$



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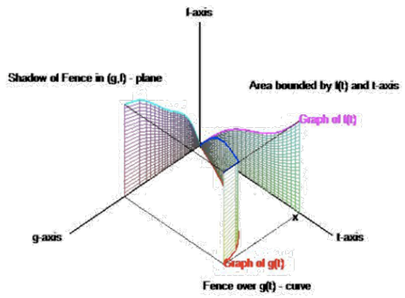
${}_0D_x^{-0.5}(t + 0.5\sin(t))$



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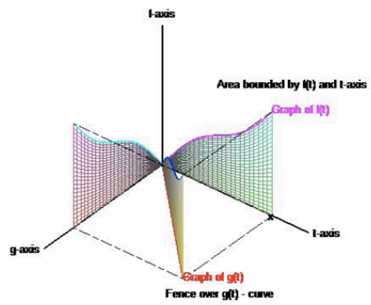
${}_0D_x^{-1}(t + 0.5\sin(t))$



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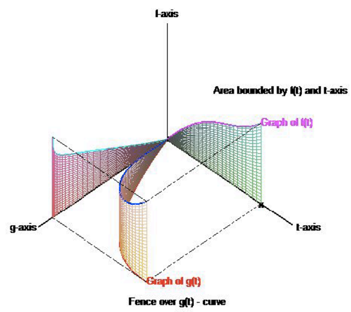
${}_0D_x^{-2.5}(t + 0.5\sin(t))$



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Fractional differential equations

$$y^{(\alpha_n)}(t) = F(t, y^{(\alpha_1)}(t), y^{(\alpha_2)}(t), \ldots, y^{(\alpha_{n-1})}(t)), \quad a < t < b$$
$$(0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n, \quad n-1 < \alpha \le n)$$

$$y^{(k)}(a) = 0, \qquad k = 0, 1, \ldots, n-1$$

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- Differential equations involving fractional derivatives.
- Example: Bagley-Torvik equation of oscillatory processes with fractional damping:

$$\frac{d^2}{dt^2}y(t) + aD_t^{1.5}y(t) + by(t) = f(t)$$

- Both ODEs and PDEs.
- Linear and non-linear.
- Existence and uniqueness of solutions established.
- Analytical solutions are difficult to evaluate.
- Dedicated, elegant numerical methods exist.

Applications:

- In spite of its long history, fractional calculus was not considered eligible for any applications.
- This was due to its high complexity and lack of physical and geometric interpretation.
- Application of fractional calculus to real-world problems is only four decades old.
- Applications can be broadly categorized into:
  1. **Modeling of Systems**
  2. **Fractional-order Control**

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Application Example

Investigation of a real process governed by fractional order differential equation:

$$D_{*a}^{\alpha}y(t) = f(t, y(t)), \qquad t \geq a$$

where

D_{*a}^{α} = Caputo differential operator of order α with starting point a.

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Investigation of a real process governed by fractional order differential equation:

$$D_{*a}^{\alpha}y(t) = f(t, y(t)), \qquad t \geq a$$

where

D_{*a}^{α} = Caputo differential operator of order α with starting point a.

Frequent obstacle

State of system can only be observed at time $t = b > a$.

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Fractional Maxwell model for viscoelastic material:

$$D^{\alpha}_{*0}\sigma(t) = \tau^{-\alpha}\sigma(t) + E \cdot D^{\alpha}_{*0}\epsilon(t) \quad (1)$$

- Task: Find shear stress $\sigma(t)$ for $t \geq 0$.

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- Measurement:

$$\sigma(b) = \sigma^* \text{ with some } b > 0 \quad (2)$$

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- Task: Find shear stress $\sigma(t)$ for $t \geq 0$.
- Measurement:

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- Known data:
  - shear strain $\epsilon(t)$
  - relaxation time τ
  - order $\alpha \in (0, 1)$
  - shear modulus E

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- Task: Find shear stress $\sigma(t)$ for $t \geq 0$.
- Measurement:

$$\sigma(b) = \sigma^* \text{ with some } b > 0 \quad (2)$$

- Approach:
  - solve (1) subject to (2) on $[0, b]$ (terminal value problem)
  - compute initial value $\sigma(0)$ from this solution
  - construct initial value problem from (1) and initial condition
  - if desired, solve initial value problem on $[0, T]$ with $T > b$

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Task: Find solution to fractional order terminal value problem

$$\begin{aligned} D^{\alpha}_{*a}y(t) &= f(t, y(t)) \\ y(b) &= y^* \end{aligned}$$

for $t \in [a, b]$.

In this talk: $0 < \alpha \leq 1$
(generalization to $\alpha > 1$ requires additional terminal conditions)

Existence and Uniqueness:

General assumptions on right-hand side of differential equation:

- continuity
- boundedness
- Lipschitz condition w. r. t. second variable

Theorems:

- Uniqueness of continuous solution to terminal value problem.
- Existence of continuous solution if interval $[a, b]$ is sufficiently small.

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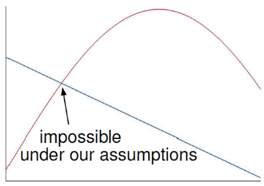
Corollary:
The graphs of two solutions to *the same differential equation* subject to *different initial or terminal conditions* never meet or cross each other.



(Di. & Ford 2012)

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Integral Equation Formulation:

Terminal value problem can be rewritten as Fredholm integral equation

$$y(t) = y^* + \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s) f(s, y(s))\, ds$$

with

$$G(t,s) = \begin{cases} -(b-s)^{\alpha-1} & \text{for } s > t, \\ (t-s)^{\alpha-1} - (b-s)^{\alpha-1} & \text{for } s \leq t. \end{cases}$$

(Di. 2010)

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Shooting Methods:

Fundamental approach:

1. Guess initial value $y(a)$
2. Solve fractional differential equation (numerically) with this initial value
3. Compare solution $y(b)$ with required terminal value y^*:
   - if $|y(b) - y^*| < \epsilon$ then accept y as approximate solution to terminal value problem,
   - if $y(b) \gg y^*$ then replace guess for initial value $y(a)$ by smaller number and go back to step 2,
   - if $y(b) \ll y^*$ then replace guess for initial value $y(a)$ by larger number and go back to step 2.

Questions:

1. Good initial guess for $y(a)$?
2. Algorithm for numerical solution of initial value problem?
3. Step size?
4. Strategy for finding improved value for $y(a)$?

Questions:

1. Good initial guess for $y(a)$?
   Use terminal value y^* (unless additional information available)
2. Algorithm for numerical solution of initial value problem?
   Use Adams-Bashforth-Moulton scheme
   (Di., Ford & Freed 2002ff.; Ford, Morgado & Rebelo 2011ff.)
3. Step size?
   Depends on required accuracy of final result
   and on quality of starting value
4. Strategy for finding improved value for $y(a)$?
   Bisection method (Ford, Morgado & Rebelo 2014)
   or Newton iteration

Step size selection strategy:

1. Error of fractional initial value problem solver = $\epsilon^{\mathrm{I}} + \epsilon^{\mathrm{N}}$:
   - ϵ^{I} = error due to incorrectly chosen initial value
   - ϵ^{N} = error introduced by numerical approximation scheme
2. Early iterations:
   - approximation of correct initial value is poor
   - ϵ^{I} is large
   - no need to have very small ϵ^{N}
   - coarse discretization of interval suffices
   - reduction of computation cost

Step size selection strategy:

1. Error of fractional initial value problem solver = $\epsilon^{\text{I}} + \epsilon^{\text{N}}$:
   - ϵ^{I} = error due to incorrectly chosen initial value
   - ϵ^{N} = error introduced by numerical approximation scheme
2. Later iterations:
   - approximation of correct initial value is good
   - ϵ^{I} is small
   - ϵ^{N} should be small as well
   - fine discretization of interval is required
   - accurate solution can be achieved

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Example problem:

$$D_{*0}^{\alpha} y(t) = \Gamma(2+\alpha)t + \frac{1}{4}\left(y(t) - w - t^{1+\alpha}\right)$$

Exact solution: $y(t) = t^{1+\alpha} + w$

Parameters:

- $\alpha = 7/10$
- $w = -3$
- $b = 12$ (rather long interval)

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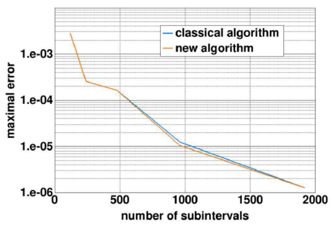
Specific step size selection strategy:

- Define number K of iterations of shooting method
- Define minimal step size h (for last iteration)
- Use step size $h_m = hK/m$ in mth iteration

Computational results:



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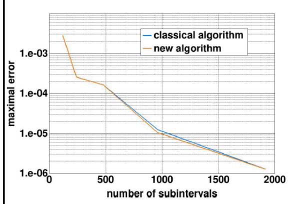

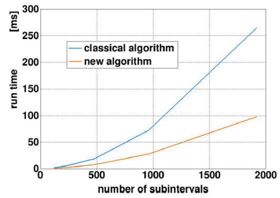

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Modeling of Physical Systems using FDEs

Diffusion:

- Normal, **Fickian** diffusion ⇒ flow of particles from high concentration to low concentration ⇒ Concentration is given by **Gaussian distribution**
- Asymptotical mean-squared displacement is a **linear function** of time,

$$\langle x^2(t)\rangle \sim t$$

- Model is given by diffusion equation

$$\frac{\partial \phi(x,t)}{\partial t} = D\frac{\partial^2 \phi(x,t)}{\partial x^2}$$

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- Some processes are an exception to this.
- Example: Photocopy machine and Laser printer. Movement of holes and electrons in the semiconductors inside them is not the normal, Gaussian diffusion.
- It is the **Anomalous diffusion**.
- Asymptotical mean-squared displacement is **not** a linear function of time,

$$\langle x^2(t)\rangle \sim t^\alpha, \quad \alpha \neq 1$$

- $\alpha < 1$ ⇒ Sub-diffusion ⇒ Slow movement of particles.
- $\alpha > 1$ ⇒ Super-diffusion ⇒ Fast movement of particles.
- Fractional diffusion equation model is

$$\frac{\partial^\alpha \phi(x,t)}{\partial t^\alpha} = D_\alpha\frac{\partial^2 \phi(x,t)}{\partial x^2}$$

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Anomalous Diffusion: Examples

- **Sub-diffusion**:
  1. Transport of holes and electrons inside the amorphous semiconductors under the electric field.
  2. Movement of contaminants in groundwater.
  3. Spread of pollutants from environmental accidents.
  4. Diffusion of proteins across cell membranes.
- **Super-diffusion**:
  1. Motion of large molecules and metal clusters across crystalline surfaces.
  2. Flight of seabirds (Albatrosses).
  3. Movement of spider monkeys.
  4. Spread of pollutants in the sea.
  5. Movement of particles inside a rapidly rotating annular tank.

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Applications of Fractional Calculus: A Panoramic View

- Viscoelastic materials.
- Polymeric materials.
- Acoustic wave propagation in inhomogeneous porous material.
- Fluid flow.
- Dynamical processes with self-similar structures.
- Dynamics of earthquakes.
- Optics.
- Geology.
- Bio-sciences.
- Medicine.

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- Electrical engineering: element Fractance.
- Economics.
- Probability and statistics.
- Astrophysics.
- Chemical engineering.
- Signal processing.
- Chaotic dynamics.
- Even fractional-order models of LOVE and EMOTIONS have been developed!!! And they are claimed to give better representation than the integer-order ODEs!!!

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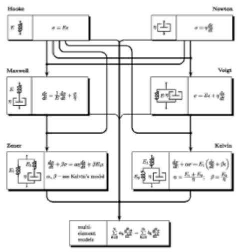
New rheological models.

New mathematical models (laws)
of deformation of viscoelastic materials.



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Impact of the process history on its state. Modelling "memory" of the process.

Fractional derivative $_0D_{t0}^{\alpha}f(t)$ is used for modelling the impact of the process history, $f(t)$ (I.e. the values of $f(t)$ for $t < t0$) on its state at time $t0$.

Hereditary properties of materials, etc.

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Dynamical processes in fractals.

Mathematical models of *dynamical* processes in fractals (self-similar structures or materials) lead to FDEs, where the order(s) of the equation(s) depends on the fractal dimension.

Porous materials, chemical reactions, diffusion, new types of electrical circuits, physiology, chaotic processes, econophysics, etc.

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Process control.

- Fractional order dynamical systems as more adequate models of real dynamical objects and processes.
- Fractional order controllers.
- Robust control.

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"Fractional-order" physics?

Hooke's law: $F = kx$

Newton's fluid: $F = kx'$

Newton's 2nd law: $F = kx''$

$\rightarrow F(t) = kx^{(\alpha)}(t)$

Diffusion–wave equation: $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}$

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The end?

No! The beginning!

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The beginning of a new stage

1695 — 1960s — You are here

static models	dynamical models	fractional order modeling
geometry, algebra	differential and integral calculus	fractional calculus

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G.W. Scott Blair (1950):

"We may express our concepts in Newtonian terms if we find this convenient but, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying."

S. Westerlund (1991):

"Expressed differently, we may say that Nature works with fractional time derivatives."

K. Nishimoto (1989):

"The fractional calculus is the calculus of the XXI century."

Confessions of a Fractional Calculus Researcher:

"As soon as I see integer-order derivatives in an equation, I replace them with the fractional ones. Then I start worrying about the motivation for the replacement."

!!!

"We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things."

Bibliography

- Igor Podlubny, *Geometric and physical interpretation of fractional integration and fraction differentiation*, Fractional Calculus and Applied Analysis, Vol.5, No. 4 (2002)
- Lokenath Debnath, Recent applications of fractional calculus to science and engineering, IJMMS, Vol. 54, pp.3413-3442 (2003)
- Keith B. Oldham, Jerome Spanier, *The Fractional Calculus; Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, (1974)
- Kenneth S. Miller, Bertram Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons (1993)

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Thank You

- Questions/comments?